# Growth Theory - Population Growth and Technological Progress

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# Introduction

The Solow model we've seen shows that capital accumulation is not enough to explain the long-run growth of economies.

- A higher savings rate leads to higher growth, but only temporarily.
- The economy will eventually reach a steady state, where growth will be zero.

In this lecture, we will introduce population growth and technological progress to the Solow model.

Let's start with population growth.

Not so fast! Before that, let me review some properties of growth rates.

# Properties of Growth rates

Let's consider a variable $X$ that grows at a rate $g_x$ and another variable $Y$ that grows at a rate $g_y$.

We want to find an expression for the growth rate of the product $Z = X \cdot Y$. Let's call it $g_z$.

By definition:

$$(1 + g_x) = \frac{X_{t+1}}{X_t}$$
$$(1 + g_y) = \frac{Y_{t+1}}{Y_t}$$
$$(1 + g_z) = \frac{Z_{t+1}}{Z_t}$$

where $t$ is a generic time period (e.g., year).

# Properties of Growth rates

Substitute $Z = X \cdot Y$ in the expression for $g_z$:

$$
\begin{aligned}
(1 + g_z) &= \frac{X_{t+1} \cdot Y_{t+1}}{X_t \cdot Y_t} \\
&= \frac{X_{t+1}}{X_t} \cdot \frac{Y_{t+1}}{Y_t} \\
&= (1 + g_x) \cdot (1 + g_y)
\end{aligned}
$$

Taking logs and using the approximation $\ln(1 + x) \approx x$ for small $x$ we have:

$$
\underbrace{\ln\left(1 + g_z\right)}_{\approx g_z} = \underbrace{\ln\left(1 + g_x\right)}_{\approx g_x} + \underbrace{\ln\left(1 + g_y\right)}_{\approx g_y} \implies
$$

$$
\boxed{g_z \approx g_x + g_y}
$$

- The growth rate of the product of two variables is (approximately) the sum of their growth rates.

# Properties of Growth rates

What if we want to find out the growth rate of the ratio $Z = X/Y$?

Rewrite this as $X = Z \cdot Y$. Now use the previous result:

$$g_x \approx g_z + g_y$$

Solving for $g_z$:

$$\boxed{g_z \approx g_x - g_y}$$

- The growth rate of the ratio of two variables is (approximately) the difference of their growth rates.

# Properties of Growth rates

What if we want to find out the growth rate of a variable $Z$ such that $Z = c \cdot X$ for some constant $c$, and $X$ grows at rate $g_x$?

$$1 + g_z = \frac{Z_{t+1}}{Z_t} = \frac{c \cdot X_{t+1}}{c \cdot X_t} = \frac{X_{t+1}}{X_t} = 1 + g_x$$

This gives us:

$$g_z = g_x$$

The growth rate of a variable that is a constant times another variable is equal to the growth rate of the other variable.

# Properties of growth rates

What if we want to find out the growth rate of the power of a variable $Z = X^\alpha$, for some constant $\alpha$?

Write:

$$1 + g_z = \frac{Z_{t+1}}{Z_t} = \frac{X_{t+1}^\alpha}{X_t^\alpha} = \left(\frac{X_{t+1}}{X_t}\right)^\alpha = (1 + g_x)^\alpha$$

Take logs and use the approximation $\ln(1 + x) \approx x$ for small $x$:

$$\ln(1 + g_z) = \alpha \ln(1 + g_x)$$

$$\boxed{g_z \approx \alpha \cdot g_x}$$

- The growth rate of the power of a variable is (approximately) the exponent times the growth rate of the variable.

Keep these properties in mind when working with growth rates!

Let's introduce population growth in the Solow model.

# Population growth in the Solow model

We will assume that the population grows at a constant rate $n$.

- We will not make a distinction between labor force and population.
- This means that $L$ grows at rate $n$ too!
- In the US, population grows at a rate of 1% per year, that means $n = 0.01\%$.

Recall the expression for a variable $L$ that grows at a constant rate $n$:

$$\boxed{L_t = L_0 \cdot (1+n)^t}$$

How does the introduction of population growth affect the steady state?
There are now three forces affecting the accumulation of capital:

- Investment $\implies$ increases capital per worker.
- Depreciation $\implies$ decreases capital per worker.
- Population growth $\implies$ decreases the amount of capital per worker. Why?
  - Let's forget depreciation for a moment.
  - From one period to the next, if we don't do anything, we are dividing the same number of machines by a larger number of workers.

# Population growth in the Solow model

The capital accumulation equation is now:

$$\boxed{\Delta k = s \cdot f(k) - nk - \delta k}$$

where:

- $s \cdot f(k)$ is investment.
- $nk$ amount necessary to equip new workers that joined the economy.
- $\delta k$ amount necessary to replace capital as it wears out.

We refer to $(n + \delta)$ as the break-even investment:

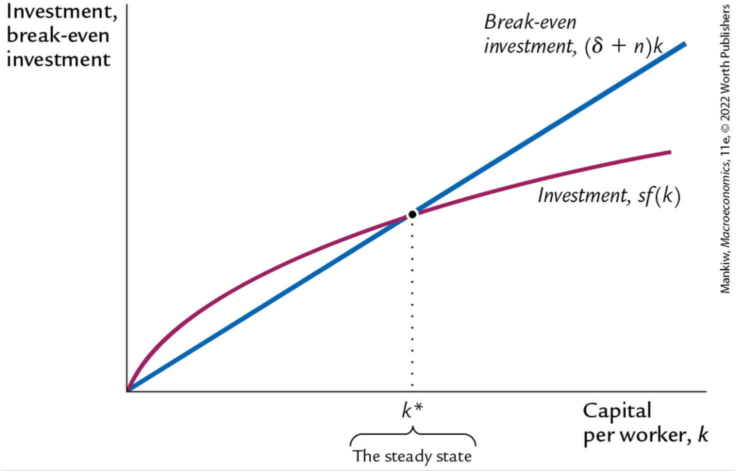
- The amount of investment needed to keep capital per worker constant.

We will again refer to steady state variables with a $*$ superscript.

As before, we define the steady state as the point where capital per worker is not changing: $\Delta k^* = 0$.

$$\boxed{s \cdot f(k^*) = (n + \delta)k^*}$$

# Population growth in the Solow model



# The effects of population growth

Note that the addition of population growth has not many effects on per capita variables!

It is just as if we had the old model with a higher "depreciation rate"!

- Instead of $\delta$, we have $(n + \delta)$!

Again, no matter what is the initial capital per worker value, the economy will converge to the steady state!

Once in the steady state, the economy will stay there forever.

The bad news: we still can't have sustained growth in per capita variables.

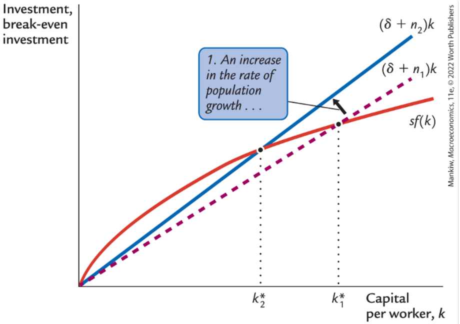
The good news: we can have sustained growth in output and capital!

- If capital per worker $k = K/L$ is constant in the steady state, that means $K$ needs to grow at the same rate as $L$.
- So $K$ grows at rate $n$.
- The same is true for output $Y$!

# The effects of population growth

What happens to capital per worker at the steady state if the rate of population growth increases from $n_1$ to $n_2$?

The effect is the same as an increase in the depreciation rate in the old model we saw!




The result: capital, output, and consumption per worker at the steady state will decrease!

# The effects of population growth

Example: Suppose the production function is a Cobb-Douglas $y = f(k) = k^{\alpha}$, population grows at rate $n$, and the depreciation rate is $\delta$. Find an expression for $y^*$.

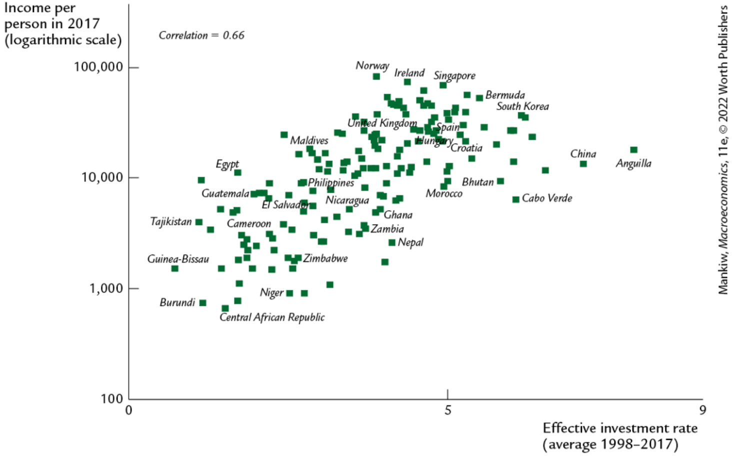
We usually refer to $s/(n+\delta)$ as the effective investment rate:

- It takes into account the savings rate and also the amount of investment needed to offset population growth and depreciation.

This result shows that countries with higher effective investment rate will have higher income per capita.

Let's check that in the real world!

# The effects of population growth




Seems good to me. What do you think?

# Alternative Perspectives on Population Growth

As we saw, high population growth rates leads to lower income per capita in the steady state.

In the model, rapid growth in the number of workers spreads the capital stock more thinly among them!

The model omits other potential effects of population growth.

We will see two alternatives perspectives:

- The Malthusian Model: emphasizes the interaction of population and natural resources.
- The Kremerian Model: emphasizes the interaction of population and technological progress.

# The Malthusian Model

The Malthusian model is named after Thomas Malthus, an English economist who lived in the 18th century.

His optimistic prediction: humans would forever live in poverty.

Population growth will outstrip the Earth's ability to produce food, leading to the impoverishment of humanity.

He argued:

> *food is necessary to the existence of man...the passion between the sexes is necessary and will remain nearly in its present state.*

This model described the world Malthus lived in, but it completely failed to predict the future:

- World population has increased about sevenfold in the last 200 years.
- Yet, average living standards are much higher.

# The Malthusian Model

Malthus neglected the effects of technological progress:

- Fertilizers, mechanized farm equipment, and new crop varieties have increased food production!
- Each farmer can now feed many more people than in Malthus's time.
- In the US today, only 1% of the population works in agriculture.

Malthus also neglected the effects of birth control:

- In some countries, the birth rate has fallen below the replacement rate.
- Over the next century, populations may be more likely to shrink than rapidly expand.

# The Kremerian Model

The Kremerian model is named after Michael Kremer, an American economist currently at University of Chicago.

Kremer argued that world population growth is a key driver of gains in prosperity.

More people means more scientists, inventor and engineers.

Evidence from very long historical periods shows that:

- As world population growth rate increased, so did the rate of growth in living standards.
- Historically, regions with larger populations have enjoyed faster growth.

# Technological Progress

We will now modify the model to include exogenous technological progress.

- We will not explain how technological progress happens!
- We will focus on the effects of technological progress on the economy.

We will modify our old production function in the following way:

$$Y = F(K, L \cdot E)$$

Here $E$ is a (very abstract, sorry) variable called the efficiency of labor.

- It captures the effect of technological progress on the productivity of labor.
- As the technology improves, each hour of work contributes more to the production of goods.
- This form of technological progress is called labor augmenting.

We still assume $L$ grows at rate $n$!

$$L_t = L_0 \cdot (1 + n)^t$$

# Technological Progress

We will call $L \times E$ the effective number of workers:

- It measures both the number of workers and the efficiency of the typical worker.

Increases in the efficiency of labor $E$ are analogous to increases in the labor force $L$:

- Suppose that advances in production methods doubles $E$ from 1990 to today.
- This means a worker today is as productive as two workers in 1990.
- In a sense, we have doubled the size of the labor force, without adding a single worker! This is good!
- We are not spreading the capital stock more thinly among workers while still getting the benefits of more workers.

# Steady State with Technological Progress

We will assume that $E$ grows at a constant rate $g$:

$$\boxed{E_t = E_0 \cdot (1+g)^t}$$

The way we choose to model technological progress makes our treatment of the steady state very simple.

It is really similar to the way we treated population growth!

We will now analyze the economy in terms of per effective worker variables.

We will refer to them with a lower case and $\sim$ superscript:

- $\tilde{k} = K/(L \cdot E)$ : Capital per effective worker.
- $\tilde{y} = Y/(L \cdot E) = f\left(\tilde{k}\right)$ : Output per effective worker.
- $\tilde{c} = C/(L \cdot E)$ : Consumption per effective worker.

Example: Find the per effective worker production function if the production function is Cobb-Douglas $Y = K^{\alpha}(L \cdot E)^{1-\alpha}$.

# Steady State with Technological Progress

The capital accumulation equation is now:

$$\boxed{\Delta\tilde{k} = s \cdot f(\tilde{k}) - (\delta + n + g)\tilde{k}}$$

where:

- $s \cdot f(\tilde{k})$ is investment per effective worker.
- $(\delta + n + g)\tilde{k}$ is the break-even investment.

The break-even investment is the amount of investment necessary to keep capital per effective worker constant:

- $\delta\tilde{k}$ is needed to replace worn-out capital.
- $n\tilde{k}$ is needed to equip new workers.
- $g\tilde{k}$ is needed to provide capital for the new effective workers.

# Steady State with Technological Progress

Again, there is one point where capital per effective worker is not changing.

This is the steady state of the model and represents the long-run equilibrium of the economy.

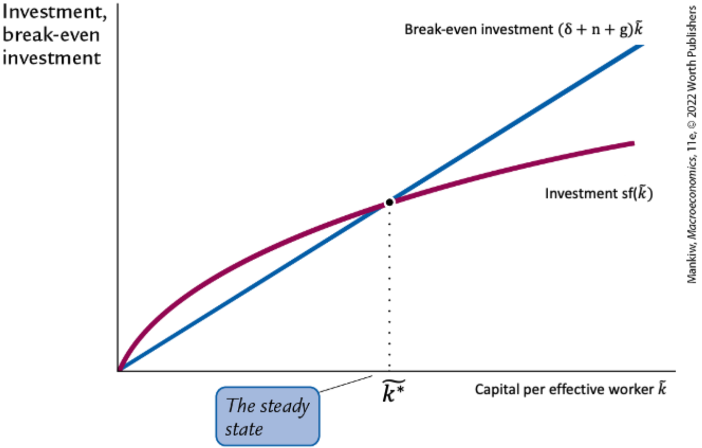
We will again refer to steady state with a $*$ superscript.

The steady state condition is given by: $\Delta \tilde{k} = 0 \implies$

$$\boxed{s \cdot f(\tilde{k}^*) = (\delta + n + g)\tilde{k}^*}$$

# Steady State with Technological Progress

Graphically:



# The effects of Technological Progress

In the steady state, we saw that capital per effective worker is constant.

Since $\tilde{y} = f(\tilde{k})$, this means that output per effective worker is also constant in the steady state.

Then consumption and investment per effective worker are also constant in the steady state, since they are a constant fraction of output.

However, we don't care about per effective worker variables. We care about per worker variables!

How do we go from per effective worker variables to per worker variables?

# The effects of Technological Progress

Just use the definition. For any instant of time $t$:

$$\tilde{k}_t = \underbrace{\frac{K_t}{L_t}}_{k_t} \cdot \frac{1}{E_t} \implies$$

$$\boxed{k_t = \tilde{k}_t \cdot E_t}$$

This expression is valid for any instant of time $t$.

Now suppose we are in the steady state at some time $t$. Then:

- Capital per effective worker is constant: $\tilde{k}_t = \tilde{k}^*$.

Substituting:

$$k_t = \tilde{k}_t \cdot E_t = \tilde{k}^* \cdot E_t$$

This means that capital per worker grows at the same rate as the efficiency of labor $E$, which is $g$. (Recall the properties of growth rates we saw earlier.)

# The effects of Technological Progress

The same is true for output and consumption per worker:

$$y_t = \tilde{y}_t \cdot E_t = \tilde{y}^* \cdot E_t$$
$$c_t = \tilde{c}_t \cdot E_t = \tilde{c}^* \cdot E_t$$

Conclusion: In the steady state

- capital per worker grows at a constant rate of $g$.
- output per worker grows at a constant rate of $g$.
- consumption per worker grows at a constant rate of $g$.

Finally, we can now have sustained growth in the long run!

Only technological progress can explain sustained growth and persistently rising living standards in the Solow model.